

An integral inequality for the invariant measure of some finite dimensional stochastic differential equation

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Dedicated to Björn Schmalfuss

Abstract

We prove an integral inequality for the invariant measure ν of a stochastic differential equation with additive noise in a finite dimensional space $H = \mathbb{R}^d$. As a consequence, we show that there exists the Fomin derivative of ν in any direction $z \in H$ and that it is given by $v_z = \langle D \log \rho, z \rangle$, where ρ is the density of ν with respect to the Lebesgue measure. Moreover, we prove that $v_z \in L^p(H, \nu)$ for any $p \in [1, \infty)$. Also we study some properties of the gradient operator in $L^p(H, \nu)$ and of his adjoint.

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1 Introduction and preliminaries

In the recent paper [DaDe14] the following inequality involving the invariant measure ν of the Burgers equation was proved

$$\left| \int_H \langle RD\varphi, z \rangle d\nu \right| \leq C_p \|\varphi\|_{L^p(H, \nu)} |z|, \quad (1.1)$$

for all $\varphi \in C_b^1(H)$, all $z \in H$ and all $p > 1$, R being a suitable negative power of the Laplace operator equipped with Dirichlet boundary conditions.

As noted in [DaDe14], by estimate (1.1) it follows that RD is closable in $L^p(H, \nu)$ for all $p > 1$. Moreover, for each $z \in H$ there exists $v_z \in L^p(H, \nu)$ such that

$$\int_H \langle RD\varphi, z \rangle d\nu = \int_H v_z \varphi d\nu, \quad \forall \varphi \in C_b^1(H). \quad (1.2)$$

Identity (1.2) implies that ν is Fomin differentiable in all directions of the range of $R(H)$ of R . We recall that if $\nu = N_Q$ (the Gaussian measure of mean 0 and covariance Q) identity (1.2) is well known in Malliavin Calculus. In this case the adjoint $(Q^{1/2}D)^*$ of $Q^{1/2}D$ is called the Skorhood operator.

The aim of the present paper is to show that the inequality (1.1), with R replaced by the identity operator, can also be proved for the invariant measures of some stochastic differential equations in $H = \mathbb{R}^d$ of the form

$$\begin{cases} dX(t) = b(X(t))dt + dW(t), \\ X(0) = x \in H, \end{cases} \quad (1.3)$$

where W is an \mathbb{R}^d -valued standard Brownian motion and b fulfills the following assumptions.

Hypothesis 1.1. (i) *There exist $\omega > 0$, $a \geq 0$ such that*

$$\langle b(x), x \rangle \leq -\omega|x|^2 + a, \quad \forall x \in \mathbb{R}^d, \quad (1.4)$$

(ii) *$b : H \rightarrow H$ is continuously differentiable and there exists $K > 0$, $N \in \mathbb{N}$ such that*

$$|b(x)| + \|b'(x)\| \leq K(1 + |x|^{2N}), \quad \forall x \in \mathbb{R}^d. \quad (1.5)$$

By (ii) it follows that b is Lipschitz continuous on bounded sets of H , whereas (i) allows to estimate $|X(t, x)|^2$ by Itô' formula; therefore existence and uniqueness of a strong solution $X(\cdot, x)$ of (1.3) is classical, see e.g. the monograph [Kr95]. We shall denote by P_t the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geq 0, x \in H, \varphi \in B_b(H) \quad (1.6)$$

For proving (1.1) we argue as in [DaDe14] starting from the elementary identity, see (3.2)

$$P_t(\langle D\varphi, h \rangle) = \langle DP_t\varphi, h \rangle - \int_0^t P_{t-s}(\langle Db \cdot h, DP_s\varphi \rangle) ds.$$

Then we prove suitable estimates for $DP_t\varphi$ and their integrals with respect to ν . These estimates require some work because, due to the polynomial growth of the derivative of b , see (1.5), we cannot exploit the classical Bismut–Elworthy–Li formula, see [El92]. To overcome this problem we shall argue as in [DaDe03], [DaDe07] and [DaDe14], introducing a suitable potential (in the present case $V(x) = K(1 + |x|^{2N})$) and the Feynman–Kac semigroup

$$S_t\varphi(x) = \mathbb{E}[\varphi(X(t, x)) e^{-\int_0^t V(X(s, x)) ds}]. \quad (1.7)$$

We shall first estimate $\langle DS_t\varphi(x), h \rangle$ then $\langle DP_t\varphi(x), h \rangle$, by taking advantage of the identity

$$P_t\varphi = S_t\varphi + \int_0^t S_{t-s}(VP_s\varphi) ds, \quad (1.8)$$

which follows from the variation of constants formula, see Section 2 below.

In Section 3 we prove that inequality (1.1) and identity (1.2) hold with $R = I$. Moreover, for any $z \in H$ we show that the Fomin derivative v_z in the direction $z \in H$ is given by $\langle D \log \rho, z \rangle$, where ρ is the density of ν with respect to the Lebesgue measure. Moreover $v_z \in L^p(H, \nu)$ for all $p \in [1, \infty)$. Finally, we prove a formula for the adjoint D^* of D and also for the elliptic operator $-\frac{1}{2}D^*D$ which can be seen as a generalisation of the Ornstein–Uhlenbeck operator.

We end this section with some notations. We set $H = \mathbb{R}^d$, $d \geq 1$ (norm $|\cdot|$, inner product $\langle \cdot, \cdot \rangle$) and denote by $L(H)$ the space of all linear bounded operators from H into H . Moreover, $C_b(H)$ is the space of all real continuous and bounded mappings $\varphi: H \rightarrow \mathbb{R}$ endowed with the sup norm

$$\|\varphi\|_\infty = \sup_{x \in H} |\varphi(x)|$$

whereas $C_b^k(H)$, $k > 1$, is the space of all real functions which are continuous and bounded together with their derivatives of order lesser than k . Finally, $B_b(H)$ will represent the space of all real, bounded and Borel mappings on H .

2 Estimates of the derivative of the transition semi-group

Let us start by giving an estimate of $\mathbb{E}(|X(t, x)|^{2m})$, $m \in \mathbb{N}$. The following lemma is standard, we shall give some details of the proof for the reader's convenience.

Lemma 2.1. *Assume Hypothesis 1.1(i). Then for any $m \in \mathbb{N}$ there exists $a_m > 0$ such that*

$$\mathbb{E}[|X(t, x)|^{2m}] \leq e^{-2m\omega t} |x|^{2m} + a_m, \quad \forall x \in H, t \geq 0. \quad (2.1)$$

Proof. Let first consider the case $m = 1$. Then by Itô's formula, taking into account (1.4) we find

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[|X(t, x)|^2] &= 2\mathbb{E}[\langle X(t, x), b(X(t, x)) \rangle] + d \\ &\leq -2\omega \mathbb{E}[|X(t, x)|^2] + 2a + d. \end{aligned}$$

We deduce that

$$\frac{d}{dt} \mathbb{E} [|X(t, x)|^2] \leq -2\omega \mathbb{E} [|X(t, x)|^2] + 2a + d.$$

By a standard comparison result it follows that

$$\mathbb{E} [|X(t, x)|^2] \leq e^{-2\omega t} |x|^2 + a_2, \quad \forall x \in H, t \geq 0, \quad (2.2)$$

where

$$a_2 = \frac{1}{\omega} (2a + d).$$

Now let $m > 1$ and $\varphi_m(x) = |x|^{2m}$. Then we have

$$D\varphi_m(x) = 2m|x|^{2m-2} x$$

and

$$D^2\varphi_m(x) = 4m(m-1)|x|^{2m-4} x \otimes x + 2m|x|^{2m-2} I,$$

where I represents identity in H . Consequently

$$\frac{1}{2} \text{Tr} [D^2\varphi_m(x)] = m(2m-2+d)|x|^{2m-2}$$

Then again by Itô's formula we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} [|X(t, x)|^{2m}] &= 2m \mathbb{E} [|X(t, x)|^{2m-2}] \langle X(t, x), b(X(t, x)) \rangle \\ &\quad + m(2m-2+d) \mathbb{E} [|X(t, x)|^{2m-2}] \\ &\leq -2m\omega \mathbb{E} [|X(t, x)|^{2m}] + m(2a+2m-2+d) \mathbb{E} [|X(t, x)|^{2m-2}]. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E} [|X(t, x)|^{2m}] &\leq e^{-2m\omega t} |x|^{2m} \\ &\quad + m(2a+2m-2+d) \int_0^t e^{-2m\omega(t-s)} \mathbb{E} [|X(s, x)|^{2m-2}] ds. \end{aligned}$$

The conclusion follows easily by recurrence. \square

Now we are going to prove an estimate for the derivative $D_x X(t, x)h$, which we denote by $\eta^h(t, x)$, $h \in H$. As well known $\eta^h(t, x)$ is a solution to the random equation

$$\begin{cases} \frac{d}{dt} \eta^h(t, x) = b'(X(t, x)) \cdot \eta^h(t, x), \\ \eta^h(0, x) = h \end{cases} \quad (2.3)$$

Lemma 2.2. *Assume Hypothesis 1.1. Then the following estimate holds*

$$|\eta^h(t, x)| \leq e^{K \int_0^t (1 + |X(s, x)|^{2N}) ds} |h|, \quad t \geq 0, \quad x, h \in H. \quad (2.4)$$

Proof. By (2.3) we deduce, taking into account (1.5), that

$$\frac{1}{2} \frac{d}{dt} |\eta^h(t, x)|^2 = \langle b'(X(t, x)) \cdot \eta^h(t, x), \eta^h(t, x) \rangle \leq K(1 + |X(t, x)|^{2N}) |\eta^h(t, x)|^2.$$

So, the conclusion follows from Gronwall's lemma. \square

Now we are going to estimate of $D_x P_t \varphi$.

2.1 Pointwise estimate

As we said in the introduction, we cannot estimate $D_x P_t \varphi$ for $\varphi \in C_b(H)$ using the Bismut–Elworthy–Li formula see [El92], because we do not know whether the expectation on the right hand side of (2.4) does exist. For this reason, we introduce the potential

$$V(x) = K(1 + |x|^{2N}), \quad x \in H$$

and the Feynman–Kac semigroup

$$S_t \varphi(x) = \mathbb{E}[\varphi(X(t, x)) e^{-\int_0^t V(X(s, x)) ds}].$$

We recall that the Bismut–Elworthy–Li formula generalises to S_t , see [DaZa97]. In fact for all $\varphi \in C_b(H)$, setting

$$\beta(t) = \int_0^t V(X(s, x)) ds,$$

the following identity holds

$$\begin{aligned} \langle DS_t \varphi(x), h \rangle &= \frac{1}{t} \mathbb{E} \left[\varphi(X(t, x)) e^{-\beta(t)} \int_0^t \langle \eta^h(s, x), dW(s) \rangle \right] \\ &- \mathbb{E} \left[\varphi(X(t, x)) e^{-\beta(t)} \int_0^t \left(1 - \frac{s}{t} \right) \langle V'(X(s, x)), \eta^h(s, x) \rangle \right] ds \quad (2.5) \\ &=: I_1(\varphi, x, h, t) + I_2(\varphi, x, h, t) = I_1 + I_2. \end{aligned}$$

We shall first estimate $\langle DS_t \varphi(x), h \rangle$, then $\langle DP_t \varphi(x), h \rangle$. In the latter case, we take advantage of the identity

$$P_t \varphi = S_t \varphi + \int_0^t S_{t-s} (V P_s \varphi) ds,$$

which follows from the variation of constants formula; in fact, denoting by \mathcal{L} and \mathcal{H} the infinitesimal generators of P_t and S_t respectively, it holds

$$\mathcal{L} = \mathcal{H} + V.$$

Lemma 2.3. *Let $\varphi \in C_b(H)$, $t \geq 0$, $x \in H$. Then for $p > 1$, there exists a constant $C_p > 0$ such that*

$$|D_x S_t \varphi(x)| \leq C_p (1 + t^{-1/2}) (1 + |x|^{2N-1}) [\mathbb{E}(\varphi^p(X(t, x)))]^{1/p}. \quad (2.6)$$

Proof. We start by estimating I_1 . By Hölder's inequality with exponents $p, q = \frac{p}{p-1}$ we have

$$\begin{aligned} |I_1| &\leq \frac{1}{t} [\mathbb{E}(\varphi^p(X(t, x)))]^{1/p} \left[\mathbb{E} \left(e^{-q\beta(t)} \left| \int_0^t \langle \eta^h(s, x), dW(s) \rangle \right|^q \right) \right]^{1/q} \\ &=: \frac{1}{t} [\mathbb{E}(\varphi^p(X(t, x)))]^{1/p} [\mathbb{E}(|z(t)|^q)]^{1/q}, \end{aligned} \quad (2.7)$$

where

$$z(t) = e^{-\beta(t)} \int_0^t \langle \eta^h(s, x), dW(s) \rangle, \quad t \geq 0. \quad (2.8)$$

We now apply Itô's formula to $g(z(t))$ where $g(r) = |r|^q$, $r \in \mathbb{R}$. Since

$$g'(r) = q|r|^{q-2}r, \quad g''(r) = q(q-1)|r|^{q-2},$$

and

$$\begin{aligned} dz(t) &= -\beta'(t)e^{-\beta(t)} \int_0^t \langle \eta^h(s, x), dW(s) \rangle ds + e^{-\beta(t)} \langle \eta^h(t, x), dW(t) \rangle \\ &= -\beta'(t)z(t) + e^{-\beta(t)} \langle \eta^h(t, x), dW(t) \rangle, \end{aligned}$$

we find

$$\begin{aligned} d|z(t)|^q &= q|z(t)|^{q-2}z(t)(-\beta'(t)z(t) + e^{-\beta(t)} \langle \eta^h(t, x), dW(t) \rangle) \\ &\quad + \frac{1}{2} q(q-1)|z(t)|^{q-2} e^{-2\beta(t)} |\eta^h(t, x)|^2 dt. \end{aligned}$$

Integrating from 0 to t , yields

$$\begin{aligned} |z(t)|^q &= -q \int_0^t |z(s)|^q \beta'(s) ds \\ &\quad + q \int_0^t |z(s)|^{q-2} z(s) e^{-\beta(s)} \langle \eta^h(s, x), dW(s) \rangle \\ &\quad + \frac{1}{2} q(q-1) \int_0^t e^{-2\beta(s)} |z(s)|^{q-2} |\eta^h(s, x)|^2 ds. \end{aligned} \quad (2.9)$$

Neglecting the negative first term in the previous identity and taking expectation, we find

$$\begin{aligned}
& \mathbb{E} \left(\sup_{r \in [0, t]} |z(r)|^q \right) \\
& \leq q \mathbb{E} \left(\sup_{r \in [0, t]} \left| \int_0^r e^{-\beta(s)} |z(s)|^{q-2} z(s) \langle \eta^h(s, x), dW(s) \rangle \right| \right) \\
& \quad + \frac{1}{2} q(q-1) \mathbb{E} \left(\int_0^t e^{-2\beta(s)} |z(s)|^{q-2} |\eta^h(s, x)|^2 ds \right) \\
& =: A_1 + A_2.
\end{aligned} \tag{2.10}$$

By the Burkholder inequality we have, taking into account Lemma 2.1

$$\begin{aligned}
A_1 & \leq 3q \mathbb{E} \left[\left| \int_0^t e^{-2\beta(s)} |z(s)|^{2(q-1)} |\eta^h(s, x)|^2 ds \right|^{1/2} \right] \\
& \leq 3q \mathbb{E} \left[\sup_{r \in [0, t]} |z(r)|^{q-1} \left(\int_0^t e^{-2\beta(s)} |\eta^h(s, x)|^2 ds \right)^{1/2} \right] \\
& \leq 3qt^{1/2} \mathbb{E} \left[\sup_{r \in [0, t]} |z(r)|^{q-1} \right] |h|.
\end{aligned} \tag{2.11}$$

By Hölder's inequality with exponents $q, \frac{q}{q-1}$, it follows that

$$A_1 \leq 3qt^{1/2} |h| \left[\mathbb{E} \left(\sup_{r \in [0, t]} |z(r)|^q \right) \right]^{\frac{q-1}{q}}. \tag{2.12}$$

Now by the Young inequality

$$ab \leq \frac{1}{u} a^u + \frac{1}{v} a^v, \quad a > 0, \quad b > 0, \quad \frac{1}{u} + \frac{1}{v} = 1 \tag{2.13}$$

with $u = q, v = \frac{q-1}{q}$, there exists $c_1 > 0$ such that

$$A_1 \leq \frac{1}{4} \mathbb{E} \left(\sup_{r \in [0, t]} |z(r)|^q \right) + c_1 t^{q/2} |h|^q. \tag{2.14}$$

Concerning A_2 , using again Lemma 2.1, we find

$$\begin{aligned}
A_2 &= \frac{1}{2}q(q-1)\mathbb{E}\left(\int_0^t e^{-2\beta(s)}|z(s)|^{q-2}|\eta^h(s,x)|^2 ds\right) \\
&\leq \frac{1}{2}q(q-1)\mathbb{E}\left[\left(\sup_{r\in[0,t]}|z(r)|^{q-2}\right)\int_0^t e^{-2\beta(s)}|\eta^h(s,x)|^2 ds\right] \\
&\leq \frac{1}{2}q(q-1)\mathbb{E}\left[\left(\sup_{r\in[0,t]}|z(r)|^{q-2}\right)\right]|h|^2 t.
\end{aligned}$$

By Hölder's inequality with exponents $\frac{q}{2}$, $\frac{q}{q-2}$ we have

$$A_2 \leq \frac{1}{2}q(q-1)|h|^2 \left[\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right)\right]^{\frac{q-2}{q}}.$$

By the Young inequality (2.13) with $u = \frac{q}{2}$ and $v = \frac{q}{q-2}$, it follows that there exists $c_2 > 0$ such that

$$A_2 \leq \frac{1}{4}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right) + c_2|h|^q t^{q/2}. \quad (2.15)$$

Taking into account (2.10), (2.14) and (2.15) we conclude that

$$\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right) \leq \frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right) + (c_1 + c_2)|h|^q t^{q/2}.$$

Therefore

$$\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right) \leq (c_1 + c_2)|h|^q t^{q/2}. \quad (2.16)$$

Finally, by (2.7) it follows that

$$I_1 \leq (c_1 + c_2)t^{-1/2}|h|[\mathbb{E}(\varphi^p(X(t,x)))]^{1/p}. \quad (2.17)$$

Now let us consider I_2 , and write

$$I_2 \leq 2KN[\mathbb{E}[\varphi^p(X(t,x))]^{1/p}(\Lambda(t))^{1/q}, \quad (2.18)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{aligned}
\Lambda(t) &= \mathbb{E} \left[e^{-q\beta(t)} \left(\int_0^t |X(s, x)|^{2N-1} |\eta^h(s, x)| ds \right)^q \right] \\
&\leq \mathbb{E} \left[\left(\int_0^t e^{-\beta(s)} |X(s, x)|^{2N-1} |\eta^h(s, x)| ds \right)^q \right] \\
&\leq \mathbb{E} \left[\sup_{r \in [0, t]} \left(|X(r, x)|^{(2N-1)q} \right) \left(\int_0^t e^{-\beta(s)} |\eta^h(s, x)| ds \right)^q \right] \\
&\leq \mathbb{E} \left[\sup_{r \in [0, t]} \left(|X(r, x)|^{(2N-1)q} \right) \right] |h|^q t^q.
\end{aligned} \tag{2.19}$$

So

$$I_2 \leq 2KN [\mathbb{E} [\varphi^p(X(t, x))]^{1/p} \left(\mathbb{E} \left[\sup_{r \in [0, t]} \left(|X(r, x)|^{(2N-1)q} \right) \right] \right)^{1/q} |h| t. \tag{2.20}$$

Recalling finally (2.1) we see that there exists $c_3 > 0$ such that

$$\left(\mathbb{E} \left[\sup_{r \in [0, t]} \left(|X(r, x)|^{2N-1} \right) \right] \right)^q \leq c_3(1 + |x|^{2N-1}),$$

so that

$$I_2 \leq 2KN c_3(1 + |x|^{2N-1}) [\mathbb{E} [\varphi^p(X(t, x))]^{1/p} |h| t. \tag{2.21}$$

Finally, by (2.5), (2.17) and (2.21), the conclusion follows easily. \square

2.2 The invariant measure ν

We shall denote by $\pi_{t,x}$ the law of $X(t, x)$ so that for each $\varphi \in B_b(H)$ we have

$$P_t \varphi(x) = \int_H \varphi(y) \pi_{t,x}(dy), \quad x \in H, \quad t > 0. \tag{2.22}$$

Lemma 2.4. *Assume Hypothesis 1.1(i). Then there is an invariant measure ν of P_t , moreover for all $m \in \mathbb{N}$ we have*

$$\int_H |x|^{2m} \nu(dx) \leq a_m, \tag{2.23}$$

where a_m is the constant in (2.1).

Proof. Let $r > 0$ and fix $x \in H$. Set $B_r^c = \{y \in H : |y| \geq r\}$. Then, taking into account (2.2) it follows that

$$\begin{aligned}\pi_{t,x}(B_r^c) &= \int_{\{|y| \geq r\}} \pi_{t,x}(dy) \leq \frac{1}{r^2} \int_H |y|^2 \pi_{t,x}(dy) \\ &= \frac{1}{r^2} \mathbb{E} [|X(t, x)|^2] \leq \frac{|x|^2 + a_2}{r^2}.\end{aligned}\tag{2.24}$$

Therefore by the Krylov–Bogoliubov theorem, see e.g [DaZa96], there exists a sequence $T_n \uparrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{T_n} \int_0^{T_n} \pi_{t,x} dt = \nu \quad \text{weakly},\tag{2.25}$$

where ν is an invariant measure of P_t .

Now we can prove (2.23). By (2.1) we deduce

$$\int_H |y|^{2m} \pi_{t,x}(dy) \leq e^{-m\omega t} |x|^{2m} + a_m, \quad \forall x \in H, t \geq 0.\tag{2.26}$$

It follows that for any $\epsilon > 0$

$$\int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \pi_{t,x}(dy) \leq e^{-m\omega t} |x|^{2m} + a_m, \quad \forall x \in H, t \geq 0.\tag{2.27}$$

Consequently integrating both sides with respect to t over $[0, T_n]$ and dividing by T_n , yields

$$\frac{1}{T_n} \int_0^{T_n} dt \int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \pi_{t,x}(dy) \leq \frac{1}{m\omega T_n} (1 - e^{-m\omega T_n}) |x|^{2m} + a_m,\tag{2.28}$$

for all $x \in H$, $t \geq 0$. Finally, letting $n \rightarrow +\infty$ and taking into account (2.25), we find

$$\int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \nu(dy) \leq a_m$$

and the conclusion follows letting ϵ tend to 0. □

2.3 Integral estimates

Let us start with an estimate of $\int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx)$.

Lemma 2.5. *Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$. Then there is $C_1 > 0$ such that*

$$\left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq C_1 (1 + t^{-1/2}) \|\varphi\|_{L^p(H, \nu)} \|h\|_{L^q(H, \nu)}.\tag{2.29}$$

Proof. Taking into account (2.6) we have

$$\begin{aligned} \left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| &\leq C_p (1 + t^{-1/2}) \\ &\times \int_H (1 + |x|^{2N-1}) [(|P_t \varphi^p(x)|)]^{1/p} |h(x)| \nu(dx). \end{aligned} \quad (2.30)$$

Let

$$\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q},$$

then by the triple Hölder inequality with exponents r, p, q we have, taking into account the invariance of ν ,

$$\begin{aligned} \left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| &\leq c (1 + t^{-1/2}) \\ &\times \left[\int_H (1 + |x|^{N-1})^r \nu(dx) \right]^{1/r} \left(\int_H |P_t \varphi^p(x)| \nu(dx) \right)^{1/p} \|h\|_{L^q(H, \nu)} \\ &\leq c (1 + t^{-1/2}) \left[\int_H (1 + |x|^{N-1})^r \nu(dx) \right]^{1/r} \|\varphi\|_{L^p(H, \nu)} \|h\|_{L^q(H, \nu)}. \end{aligned} \quad (2.31)$$

The conclusion follows from (2.23). \square

Now we are ready to estimate $\int_H \langle D_x P_t \varphi(x), h(x) \rangle \nu(dx)$. We start from the identity

$$P_t \varphi(x) = S_t \varphi(x) + K \int_0^t S_{t-s} (1 + |x|^{2N}) P_s \varphi(x) ds, \quad (2.32)$$

from which

$$\begin{aligned} &\langle D_x P_t \varphi(x), h(x) \rangle \\ &= \langle D_x S_t \varphi(x), h(x) \rangle + K \int_0^t \langle D_x S_{t-s} ((1 + |x|^{2N}) P_s \varphi)(x), h(x) \rangle ds. \end{aligned} \quad (2.33)$$

Proposition 2.6. *Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$. Then there is C_p^1 such that*

$$\left| \int_H \langle D_x P_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq C_p^1 (1 + t^{-1/2}) \|\varphi\|_{L^p(H, \nu)} \|h\|_{L^q(H, \nu)}. \quad (2.34)$$

Proof. The first term of (2.33) is bounded by (2.29). Let us estimate the second one. Again by (2.29) we have

$$\begin{aligned} &\left| \int_0^t \int_H \langle D_x S_{t-s} ((1 + |x|^{2N}) P_s \varphi), h(x) \rangle ds \right| \nu(dx) \\ &\leq C_1 \int_0^t (1 + (t-s)^{-\frac{1}{2}}) \|(1 + |x|^{2N}) P_s \varphi\|_{L^p(H, \nu)} \|h\|_{L^q(H, \nu)} ds \end{aligned} \quad (2.35)$$

Now let us chose $\epsilon > 0$ such that

$$\frac{1}{p+\epsilon} + \frac{1}{q} < 1.$$

Then by Hölder's inequality with exponents $\frac{p+\epsilon}{\epsilon}$ and $\frac{p+\epsilon}{p}$ it follows that

$$\begin{aligned} \|(1 + |x|^{2N}) P_s \varphi\|_{L^p(H, \nu)}^p &= \int_H (1 + |x|^{2N})^p (P_s \varphi)^p d\nu \\ &\leq \left(\int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{\epsilon}} d\nu \right)^{\frac{\epsilon}{p+\epsilon}} \left(\int_H (P_s \varphi)^{p+\epsilon} d\nu \right)^{\frac{p}{p+\epsilon}} \\ &\leq \left(\int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{\epsilon}} d\nu \right)^{\frac{\epsilon}{p+\epsilon}} \|\varphi\|_{L^{p+\epsilon}(H, \nu)}^p, \end{aligned}$$

by the invariance of ν . Now by (2.2) there exists a constant C' such that

$$\int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{\epsilon}} d\nu \leq C',$$

Therefore

$$\|(1 + |x|^{2N}) P_s \varphi\|_{L^p(H, \nu)}^p \leq (C')^{\frac{\epsilon}{p+\epsilon}} \|\varphi\|_{L^{p+\epsilon}(H, \nu)}^p. \quad (2.36)$$

Substituting in (2.35), yields

$$\begin{aligned} &\left| \int_0^t \langle D_x S_{t-s}(|x|_N^N P_s \varphi), h(x) \rangle ds \right| \\ &\leq C_1 (C')^{\frac{\epsilon}{p+\epsilon}} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) \|\varphi\|_{L^{p+\epsilon}(H, \nu)} \|h\|_{L^q(H, \nu)} ds \end{aligned} \quad (2.37)$$

Non the conclusion follows by the arbitrariness of ϵ, p, q . \square

3 The main inequality and its consequences

Theorem 3.1. *For all $p > 1$ there exists a constant $C_p > 0$ such that for all $\varphi \in L^p(H, \nu)$ and all $h \in H$ we have*

$$\left| \int_H \langle D_x \varphi(x), h \rangle \nu(dx) \right| \leq c \|\varphi\|_{L^p(H, \nu)} |h|. \quad (3.1)$$

Proof. Step 1. For any $\varphi \in C_b^1(H)$ and any $h \in H$ the following identity holds.

$$P_t(\langle D\varphi, h \rangle) = \langle DP_t \varphi, h \rangle - \int_0^t P_{t-s}(\langle Db \cdot Hh, DP_s \varphi \rangle) ds, \quad t > 0. \quad (3.2)$$

To prove (3.2) we consider a sequence (b_n) of mappings $H \rightarrow H$ of class C^∞ such that

- (i) $\lim_{n \rightarrow \infty} b_n(x) = b(x)$, uniformly on bounded sets of H .
- (ii) $\langle b_n(x), x \rangle \leq -\omega|x|^2 + a, \quad \forall x \in H$.

To construct (b_n) we first set

$$f_n(x) = \frac{b(x) + \omega x}{1 + n^{-1}|x|^{2N+2}} - \omega x,$$

so that

$$\langle f_n(x), x \rangle \leq -\omega|x|^2 + a, \quad \forall x \in H,$$

and f_n is sub-linear, then we regularise f_n using mollifiers.

Now we prove the identity

$$P_t^n(\langle D\varphi, h \rangle) = \langle DP_t^n \varphi, h \rangle - \int_0^t P_{t-s}(\langle Db \cdot h, DP_s^n \varphi \rangle) ds, \quad (3.3)$$

where P_t^n is the transition semigroup corresponding to b_n .

It is enough to show (3.3) for each $\varphi \in C_b^3(H)$. In such a case set $u_n(t, x) = P_t^n \varphi(x)$ and write

$$\begin{cases} D_t u_n(t, x) = \frac{1}{2} \Delta u_n(t, x) + \langle Du_n(t, x), b_n(x) \rangle, \\ u_n(0, x) = \varphi(x). \end{cases} \quad (3.4)$$

Now, taking $h \in H$ and setting

$$v_n(t, x) = \langle Du_n(t, x), h \rangle$$

we see, by a simple computation, that

$$\begin{cases} D_t v_n(t, x) = \frac{1}{2} \Delta v_n(t, x) + \langle Dv_n(t, x), b_n(x) \rangle \\ \quad + \langle Du_n(t, x), b'_n(x)h \rangle, \\ v_n(0, x) = \langle D\varphi(x), h \rangle. \end{cases} \quad (3.5)$$

By the variation of constants formula it follows that

$$v_n(t, x) = P_t^n(\langle D\varphi(x), h \rangle) + \int_0^t P_{t-s}^n \langle Du_n(s, x), Ah + b'_n(x)h \rangle ds, \quad (3.6)$$

which coincides with (3.3). Letting $n \rightarrow \infty$, yields (3.2).

Step 2. Conclusion.

Integrating (3.2) with respect to ν over H and taking into account the invariance of ν , yields

$$\begin{aligned} \int_H \langle D\varphi(x), h \rangle \nu(dx) &= \int_H \langle DP_t\varphi(x), h \rangle \nu(dx) \\ &- \int_H \int_0^t \langle b'(x)h, DP_s\varphi(x) \rangle ds \nu(dx) =: J_1 + J_2 \end{aligned} \quad (3.7)$$

Setting and $t = 1$ we deduce

$$|J_1| \leq \left| \int_H \langle D_x P_t \varphi(x), h \rangle \nu(dx) \right| \leq 2C_p^1 \|\varphi\|_{L^p(H, \nu)} |h|. \quad (3.8)$$

Concerning J_2 we have by (2.34) and taking into account (1.5)

$$\begin{aligned} |J_2| &\leq \int_0^t \int_H C_p^1 (1 + (t-s)^{-1/2}) \|\varphi\|_{L^p(H, \nu)} \|b'(\cdot)h\|_{L^q(H, \nu)} ds \\ &\leq K \int_0^t \int_H C_p^1 (1 + (t-s)^{-1/2}) \|\varphi\|_{L^p(H, \nu)} \|(1 + |x|^{2N})\|_{L^q(H, \nu)} ds |h|. \end{aligned} \quad (3.9)$$

Finally, recalling (2.23) and setting $t = 1$ the conclusion follows. \square

3.1 Consequences of the integral inequality (3.1)

The following result can be proved exactly as in [DaDe14], replacing R by I so, we omit the proof.

Proposition 3.2. *Assume Hypothesis 1.1 and let ν be the invariant measure of problem (1.3). Then for any $p > 1$ the gradient*

$$D : C_b^1(H) \subset L^p(H, \nu) \rightarrow L^p(H, \nu; H), \quad \varphi \rightarrow D\varphi,$$

is closable.

For any $p > 1$ we shall denote by D_p the closure of D and by D_p^* the adjoint operator of D_p . D_p is a mapping

$$D_p : D(D_p) \subset L^p(H, \nu) \rightarrow L^p(H, \nu; H)$$

and D_p^* is a mapping

$$D_p^* : D(D_p^*) \subset L^q(H, \nu; H) \rightarrow L^q(H, \nu),$$

where $q = \frac{1}{1-p}$. We have obviously

$$\int_H \langle D_p \varphi, F \rangle d\nu = \int_H \varphi D_p^*(F) d\nu, \quad (3.10)$$

for any $\varphi \in D(D_p)$ and any $F \in D(D_p^*)$. We recall that $F \in D(D_p^*)$ if and only if there exists a positive constant K_F such that

$$\left| \int_H \langle D\varphi, F \rangle d\nu \right| \leq K_F \|\varphi\|_{L^p(H, \nu)}, \quad \forall \varphi \in C_b^1(H). \quad (3.11)$$

In this case we have

$$\|D_p^*(F)\|_{L^q(H, \nu)} \leq K_F. \quad (3.12)$$

If no confusion may arise we shall omit sub-indices p in D_p and D_p^* .

Proposition 3.3. *For any $z \in H$ there is $v_z \in L^q(H, \nu)$ for all $q \in [1, +\infty)$ such that*

$$\int_H \langle D\varphi, z \rangle d\nu = \int_H v_z \varphi d\nu. \quad (3.13)$$

Proof. Let $z \in H$ and set $F_z(x) = z$, $\forall x \in H$. Then by (3.1) it follows that

$$\left| \int_H \langle D\varphi, F_z \rangle d\nu \right| \leq C_{1,p} \|\varphi\|_{L^p(H, \nu)} |z| \quad (3.14)$$

This implies $F_z \in D(D_p^*)$ and $\|D_p^*(F_z)\|_{L^q(H, \nu)} \leq C_{1,p}|z|$. Setting $D_q^*(F_z) = v_z$, identity (3.13) follows. \square

Remark 3.4. By Proposition 3.3 ν possesses the Fomin derivative of ν at the direction z which is given precisely by v_z and so, it belongs to $L^q(H, \nu)$ for all $q \in [1, \infty)$,

Now we are going to identify v_z .

Proposition 3.5. *Assume Hypothesis 1.1. Then for any $z \in H$ we have $v_z = \langle D \log \rho, z \rangle$, where ρ is the density of ν with respect to the Lebesgue measure on \mathbb{R}^d . Therefore $\langle D \log \rho, z \rangle$ belong to $L^p(H, \nu)$ for any $p \in [1, +\infty)$.*

Proof. First notice that by (3.13) it follows in particular that

$$\left| \int_H \langle D\varphi, z \rangle d\nu \right| \leq \|v_z\|_{L^1(H, \nu)} \|\varphi\|_\infty. \quad (3.15)$$

Therefore, by an argument due to Malliavin, ν has a density ρ with respect to the Lebesgue measure on \mathbb{R}^d with $\rho \in L^{\frac{d}{d-1}}(\mathbb{R}^d)$, see [Nu95].

To prove the last statement, we write (3.13) as

$$\int_H \langle D\varphi, z \rangle \rho dx = \int_H \varphi v_z \rho dx.$$

This implies in the sense of distributions that

$$v_z = \langle D \log \rho, z \rangle.$$

Now the conclusion follows from Proposition 3.3. \square

Remark 3.6. The fact that ν has a density ρ with respect to the Lebesgue measure, together with several properties of ρ have already been proved in [MePaRh05], [BoKrRo01] and [BoKrRo05].

Let us finally study some properties of operators D^* and D^*D .

Proposition 3.7. *Let*

$$F(x) = \sum_{h=1}^d f_h(x) e_h, \quad x \in H, \quad (3.16)$$

where (e_1, \dots, e_d) is an orthonormal basis in H and $f_h \in C_b^1(H)$, $h = 1, \dots, d$. Then F belongs to the domain of D^* and it results

$$D^*(F) = -\operatorname{div} F + \sum_{h=1}^d v_{e_h} f_h. \quad (3.17)$$

Moreover, if $\varphi \in C_b^2(H)$ we have

$$-\frac{1}{2} D^* D(\varphi) = \frac{1}{2} \Delta \varphi - \frac{1}{2} \sum_{h=1}^d v_{e_h} D_h \varphi. \quad (3.18)$$

Proof. Write

$$\begin{aligned} \int_H \langle D\varphi, F \rangle d\nu &= \sum_{h=1}^d \int_H D_h \varphi f_h d\nu \\ &= \sum_{h=1}^d \int_H D_h(\varphi f_h) d\nu - \sum_{h=1}^d \int_H \varphi D_h f_h d\nu \end{aligned} \quad (3.19)$$

Since, in view of (3.13)

$$\int_H D_h(\varphi f_h) d\nu = \int_H \varphi v_{e_h} f_h d\nu,$$

(3.17) follows. Now (3.18) follows as well setting $F = D\varphi$ in (3.17). □

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